UNIT 3

Matter Waves and the Double Slit

De Broglie's wavelength, electron diffraction, and the double slit with one electron at a time

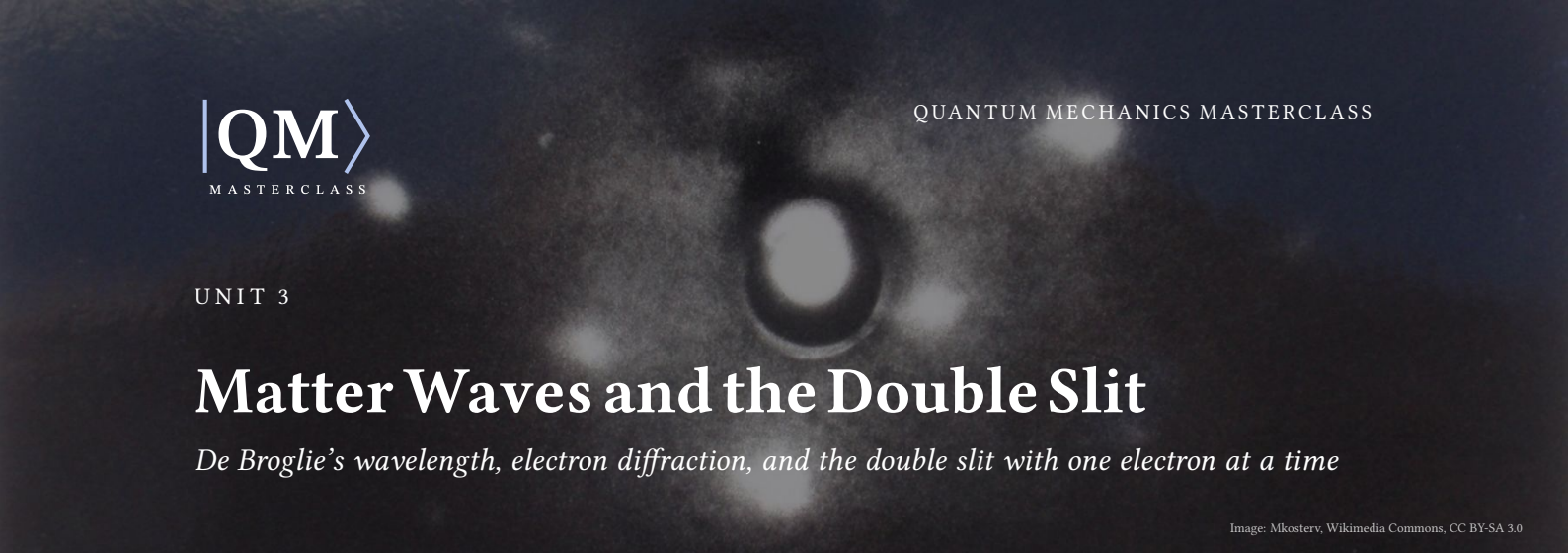


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By the end of this unit, I should be able to answer these questions:

- What is the de Broglie wavelength of an electron accelerated through 54 volts, and why is it close to the spacing of atoms in a nickel crystal?
- Why do we never notice the wave nature of a tennis ball?
- What did Davisson and Germer observe in 1927 when they scattered electrons from a nickel crystal?
- How can bright and dark bands appear when electrons pass through a double slit one at a time?
- What happens to the pattern when we find out which slit each electron passed through?
- Why can a molecule of 10,000 atomic mass units still interfere, and what limits the resolution of an electron microscope?

Unit 2 ended with light behaving as a stream of particles, each carrying a momentum $p = h/\lambda$. In 1924 a doctoral student in Paris turned this sentence around and proposed that every particle with momentum p is accompanied by a wave of wavelength h/p. Within three years the wave had been seen in the laboratory, and within sixty years electrons were filmed building up an interference pattern one at a time. This unit follows that path from the hypothesis to the experiments, works out the wavelengths of electrons, neutrons, atoms, molecules, and a tennis ball, and ends with the two places where matter waves are used every day, the electron microscope and the molecule interferometer.

1 De Broglie's hypothesis

Louis de Broglie (1892-1987) started from a symmetry. Light had been known to be a wave for a century, and Unit 2 showed that it also comes in photons with energy $E = hf$ and momentum $p = h/\lambda$. If a wave can carry the properties of a particle, de Broglie asked, why should a particle not carry the properties of a wave? He proposed that the same two relations connect the momentum and energy of any moving body, an electron as much as a photon, to a wavelength and a frequency. The **de Broglie wavelength** of a particle is the wavelength that goes with its momentum.

KEY EQUATION The de Broglie relation

$$\lambda = \frac{h}{p} = \frac{h}{mv} \qquad \text{and, for a particle of kinetic energy } K, \qquad \lambda = \frac{h}{\sqrt{2mK}}$$

λ is the wavelength of the matter wave, $p = mv$ the momentum of a particle of mass m and speed v, and $h = 6.63\times10^{-34}$ J s Planck's constant. The second form uses $K = p^2/2m$ and holds for speeds much smaller than the speed of light. For an electron accelerated through a voltage V it becomes $\lambda = h/\sqrt{2m_e eV}$, which is $1.226\,\text{nm}/\sqrt{V}$ with V taken as a bare number of volts.

The second form follows from the first by solving $K = \frac{1}{2}mv^2 = p^2/2m$ for p. An electron that has been accelerated through a voltage V has gained the kinetic energy $K = eV$, so its wavelength is $h/\sqrt{2m_e eV}$, and with the constants put in this is 1.226 nm divided by the square root of the voltage in volts. A third form is useful for atoms and neutrons in a gas at temperature T, whose mean kinetic energy is $\frac{3}{2}k_B T$ with Boltzmann's constant k_B (Unit 1). Putting this energy in for K gives their typical wavelength, $\lambda = h/\sqrt{3mk_B T}$.

WORKED EXAMPLE The electron of 54 volts

Davisson and Germer, whose experiment Section 2 describes, used electrons accelerated through 54 V. Their kinetic energy is $54\,\text{eV} = 8.65 \times 10^{-18}$ J (one electronvolt is 1.602×10^{-19} J), their momentum is

$$p = \sqrt{2m_e K} = \sqrt{2 \times 9.109 \times 10^{-31}\,\text{kg} \times 8.65 \times 10^{-18}\,\text{J}} = 3.97 \times 10^{-24}\,\text{kg m s}^{-1},$$

and the de Broglie relation gives

$$\lambda = \frac{h}{p} = \frac{6.626 \times 10^{-34}\,\text{J s}}{3.97 \times 10^{-24}\,\text{kg m s}^{-1}} = 1.67 \times 10^{-10}\,\text{m}.$$

The wavelength is $\underline{\underline{\lambda = 0.167\,\text{nm}}}$, and the short form $1.226/\sqrt{54} = 0.167$ gives the same number. This is about 3,000 times shorter than the wavelength of visible light, which is about 500 nm, and close to the distance between neighbouring atoms in a metal. That is why a crystal, and not an ordinary ruled grating (a glass plate with thousands of fine parallel lines scratched into it), was the right instrument to look for the wave.

WORKED EXAMPLE A thermal neutron and a tennis ball

A neutron in a gas at room temperature, $T = 300\,\text{K}$, has the mean kinetic energy $\frac{3}{2}k_B T = 38.8\,\text{meV}$ (millielectronvolts, thousandths of an electronvolt). With the neutron mass $m_n = 1.675 \times 10^{-27}$ kg its wavelength is

$$\lambda = \frac{h}{\sqrt{3m_n k_B T}} = \frac{6.626 \times 10^{-34}}{\sqrt{3 \times 1.675 \times 10^{-27} \times 1.381 \times 10^{-23} \times 300}}\,\text{m} = 1.45 \times 10^{-10}\,\text{m},$$

so a thermal neutron has $\underline{\underline{\lambda = 0.145\,\text{nm}}}$, again the size of the spacing of atoms in a solid. Neutrons slowed down to room temperature in a reactor are for this reason a standard probe of crystal structures.

A tennis ball of mass 57 g served at $50\,\text{m s}^{-1}$, which is 180 kilometres per hour, has the momentum $p = 0.057\,\text{kg} \times 50\,\text{m s}^{-1} = 2.85\,\text{kg m s}^{-1}$ and the wavelength

$$\lambda = \frac{6.626 \times 10^{-34}\,\text{J s}}{2.85\,\text{kg m s}^{-1}} = 2.3 \times 10^{-34}\,\text{m}.$$

The wavelength of the ball is $\underline{\underline{2.3 \times 10^{-34}\,\text{m}}}$, which is nearly 10^{19} times smaller than the diameter of a proton, itself only about 1.7×10^{-15} m. No slit, edge, or obstacle in the universe is small enough to make such a wave spread out noticeably, and that is the whole reason why nobody has ever seen a tennis ball diffract.

Table 1 collects these numbers with a few more. Two patterns run through the table. The first is that heavier particles at the same speed have shorter wavelengths, because the momentum grows with the mass. The second is that the wavelength falls only with the square root of the kinetic energy, so a thousandfold increase in energy shortens it by a factor of about 32, the square root of 1,000. For everything we can hold in a hand the wavelength is unmeasurably small, and for electrons, neutrons, and atoms it is about the size of an atom. **Diffraction**, the spreading and interference of a wave at an obstacle, is only noticeable when the obstacle is not much larger than the wavelength, so the wave nature of matter can only show itself at the scale of atoms.

Particle	Kinetic Energy or Speed	Wavelength
Electron	54 eV	0.167 nm
Electron	300 keV (electron microscope)	1.97 pm
Neutron	thermal, 300 K	0.145 nm
Helium atom	thermal, 300 K	72.9 pm
Fullerene C_{60}, 720 u	200 m s^{-1}	2.77 pm
Tennis ball, 57 g	50 m s^{-1}	2.3×10^{-34} m

Table 1. De Broglie wavelengths of particles from the electron to a tennis ball. The thermal values use the mean kinetic energy $\frac{3}{2}k_B T$; the value for the 300 keV electron includes the relativistic correction of Section 6. The mass unit u is the atomic mass unit, one twelfth of the mass of a carbon-12 atom, and 1 pm (picometre) is 10^{-12} m.

CHECK YOUR UNDERSTANDING Question 1

An electron and a proton have the same kinetic energy. Which of them has the longer de Broglie wavelength, and by roughly what factor?

(A) The proton, by a factor of about 43

(B) Both have the same wavelength, since the energy is the same

(C) The electron, by a factor of about 43

(D) The electron, by a factor of about 1,840

The solution is at the end of the unit.

EXCURSION ***A thesis and a letter to Einstein***

De Broglie came to physics late. He had studied history, served in the army as a radio operator on the Eiffel Tower during the First World War, and worked in the private laboratory of his older brother Maurice, an experimental physicist who studied X-rays. In September and October 1923 he published three short notes in the Comptes rendus of the French Academy of Sciences with the idea of matter waves, and in 1924 he collected them in his doctoral thesis, Recherches sur la théorie des quanta (Research on the theory of quanta), which he defended at the Sorbonne on 25 November 1924.

The thesis contained a striking result. If an electron on one of Bohr's orbits around the hydrogen nucleus is a wave, the wave must fit around the orbit a whole number of times, and this condition reproduces exactly the quantization rule that Bohr had postulated in 1913. Unit 7 works this out. The examiners were impressed by the mathematics and unsure about the physics, and Paul Langevin (1872-1946), one of them, sent a copy to Albert Einstein (1879-1955) to ask what he made of it. Einstein replied in December 1924 that de Broglie had, in his words, lifted a corner of the great veil, and he used the idea a few weeks later in his own work on gases. Schrödinger read de Broglie's thesis in 1925, after Einstein's paper on the quantum gas had drawn attention to it, and built his wave equation on it the following winter (Unit 4). In 1927 the waves were found in the laboratory, and in 1929 de Broglie received the Nobel Prize in Physics for the discovery of the wave nature of electrons, a rare case of a Nobel Prize awarded for work done in a doctoral thesis.

2 Electron diffraction

A wave shows itself by diffraction and interference, and for a wavelength of a tenth of a nanometre the only gratings with a fine enough spacing are crystals, in which the atoms sit in rows about 0.2 to 0.4 nm apart. Clinton Davisson (1881-1958) at the Western Electric laboratories in New York, which became the Bell Telephone Laboratories in 1925, had been firing slow electrons at nickel targets since 1921 to study how electrons

bounce off metals, and in October 1924 Lester Germer (1896-1971) joined him in this work. In April 1925 a bottle of liquid air exploded next to their apparatus, air entered the vacuum tube, and the hot nickel target oxidized. The long heating needed to clean it turned the fine-grained metal into a few large crystals, because at high temperature the small crystal grains of a metal grow into larger ones. The scattering pattern changed completely, and from then on they worked with a single crystal cut to expose one flat face. In the summer of 1926 Davisson learned on a visit to England that theorists expected electrons to diffract, and in January 1927 the two found what they were looking for.

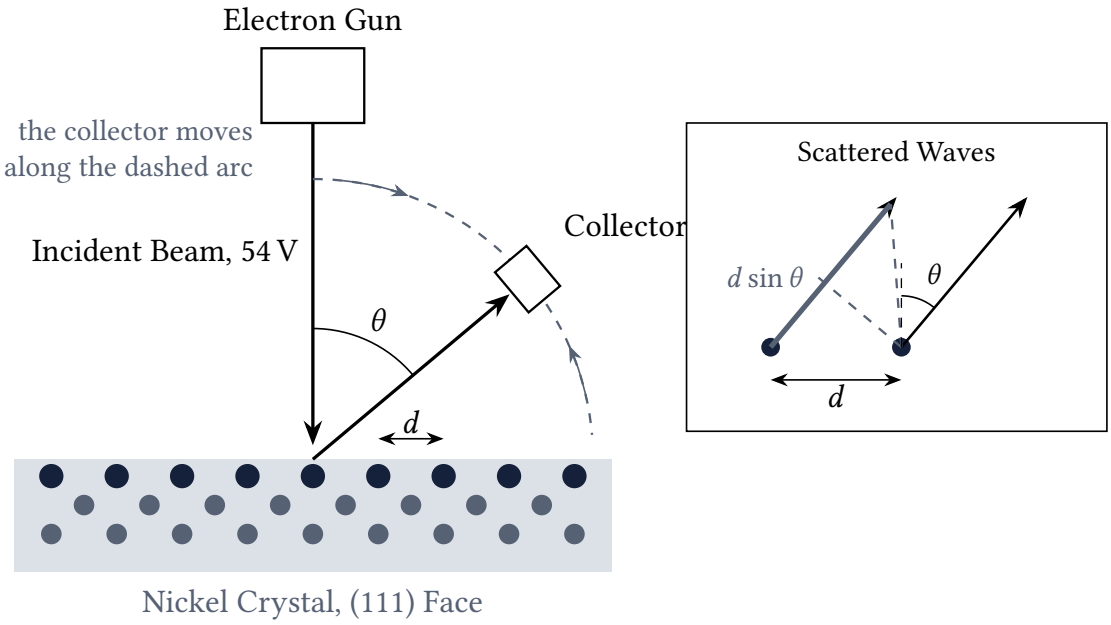


Figure 1. The Davisson-Germer experiment. A beam of electrons falls at right angles on to the face of a nickel crystal, and a collector that moves along an arc measures how many electrons leave in each direction. The rows of atoms in the surface, a distance d apart, act as a diffraction grating; the inset shows that waves scattered by two neighbouring rows into the direction θ differ in path length by $d \sin\theta$.

Figure 1 shows the arrangement. Electrons of a chosen energy hit the crystal face at right angles, and the collector counts those that leave elastically, with their full energy, as a function of the angle θ from the incident direction. Along the surface the atoms form rows with a spacing d, and each row scatters a little of the incoming wave. In a direction θ, the waves from two neighbouring rows have travelled paths that differ by $d \sin\theta$, as the inset shows. When this difference is a whole number of wavelengths, the waves from all rows arrive in step and reinforce one another; in other directions they cancel. This is the same condition as for a ruled grating in optics.

KEY EQUATION Diffraction condition for a row of atoms

$$d \sin\theta = n\lambda, \qquad n = 1, 2, 3, \ldots$$

Waves scattered at normal incidence (the beam falling at right angles on to the surface) by rows of atoms a distance d apart reinforce one another in the directions θ for which the path difference $d \sin\theta$ is a whole multiple n of the wavelength λ. The strongest beam is the first order, $n = 1$.

WORKED EXAMPLE Where the 54 volt beam should appear

On the crystal face that Davisson and Germer used, the rows of nickel atoms are $d = 0.215\,\text{nm}$ apart.

With the wavelength $\lambda = 0.167\,\text{nm}$ of the first example, the first-order condition gives

$$\sin\theta = \frac{\lambda}{d} = \frac{0.167\,\text{nm}}{0.215\,\text{nm}} = 0.777, \qquad \theta = 51^\circ.$$

De Broglie's relation therefore predicts a diffracted beam at $\theta \approx 51^\circ$. Davisson and Germer observed the beam at 50°, which corresponds to a measured wavelength of $0.215\,\text{nm} \times \sin 50^\circ = 0.165\,\text{nm}$. The one per cent difference was understood a year later. Inside the crystal the metal attracts the electrons and speeds them up, so their wavelength there is a little shorter. The layers of atoms below the surface help to form the beam, and so the beam leaves at a slightly shifted angle. The second order, $n = 2$, would need $\sin\theta = 1.55$, which no angle satisfies, so at this voltage only one beam exists.

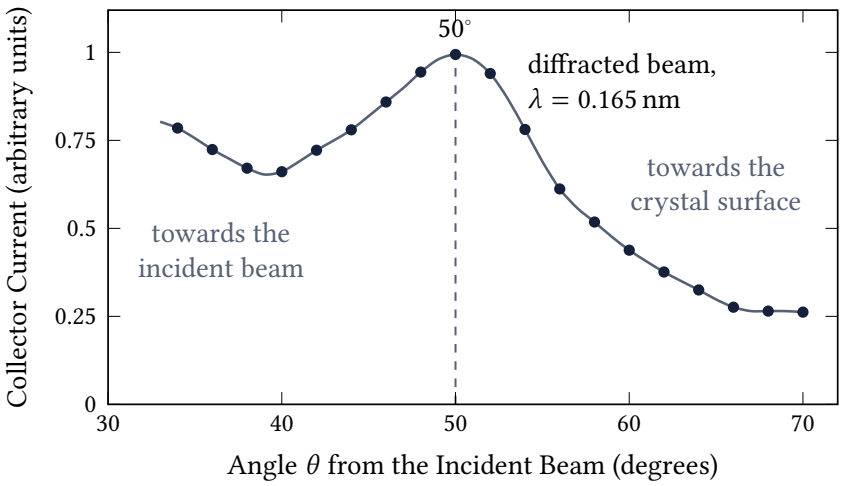


Figure 2. The measurement of 1927. The collector current for 54 V electrons, read from Figure 10 of Davisson and Germer's paper in the Physical Review of December 1927, against the angle from the incident beam. The sharp maximum at 50° is the diffracted beam, and it disappears when the voltage is changed by a few volts in either direction. The current scale is the paper's arbitrary one.

Figure 2 shows their measurement. The beam at 50° appears only when the voltage is near 54 V. At lower voltages the wavelength is longer and the beam moves to larger angles, and at higher voltages it moves toward the incident direction, exactly as the diffraction condition demands. Davisson and Germer went on to find twenty more such beams at different voltages and from different sets of rows, and nineteen of these could be compared with $\lambda = h/p$. The agreement was within the accuracy of the measurements. Their preliminary report appeared in the journal Nature in April 1927 and the full paper in December of the same year.

Two months after the Nature letter, George Paget Thomson (1892-1975) in Aberdeen reported the same effect in a completely different form. He sent fast electrons, accelerated through tens of thousands of volts, through a thin film of celluloid and later through films of metal, and on a photographic plate behind the film he found rings around the central spot. The films consist of countless tiny crystals in every orientation, so the diffracted beams leave at the same angle θ but in every direction around the beam axis, and form cones instead of single spots, exactly as X-rays do when they pass through a powder of small crystals. The radii of the rings gave the wavelength, and it agreed with de Broglie's formula. Davisson and Thomson shared the Nobel Prize in Physics 1937. Thomson's father, J. J. Thomson, had received the prize of 1906 for his work on the conduction of electricity through gases, the work in which he had identified the electron as a particle, so the father was honoured for showing that the electron is a particle and the son for showing that it is a wave.

Matter waves were soon found for heavier particles. In 1930 Immanuel Estermann (1900-1973) and Otto Stern (1888-1969) in Hamburg diffracted beams of helium atoms and hydrogen molecules from the surface of a lithium fluoride crystal, and from 1936 on, slow neutrons were diffracted by crystals as well. The technique of Davisson and Germer lives on as low-energy electron diffraction, the method shown on the title page of this unit, which is

used today to determine how atoms are arranged in the outermost layer of a surface. Electrons of 94 V, as in that picture, have a wavelength of 0.126 nm.

CHECK YOUR UNDERSTANDING Question 2

Davisson and Germer saw the first-order beam at 50 degrees with 54 volt electrons. The voltage is now raised to 216 volts, four times as much. Where does the first-order beam from the same rows of atoms appear?

(A) Near 23 degrees, closer to the incident beam

(B) Still at 50 degrees, since the rows of atoms have not changed

(C) Near 12 degrees, closer to the incident beam

(D) At an angle larger than 50 degrees, farther from the incident beam

The solution is at the end of the unit.

3 The double slit, one electron at a time

Diffraction by a crystal proved that electrons have a wavelength, but the experiment that shows what this means most clearly is older and simpler. In the first years of the nineteenth century Thomas Young (1773-1829) let light fall on two narrow slits close together and found bright and dark bands on a screen behind them. Figure 3 shows the geometry. The slits are a distance d apart, the screen is a distance L behind them, and a point P on the screen lies a height y above the axis. Light reaches P from both slits, and the two waves have travelled paths of different lengths.

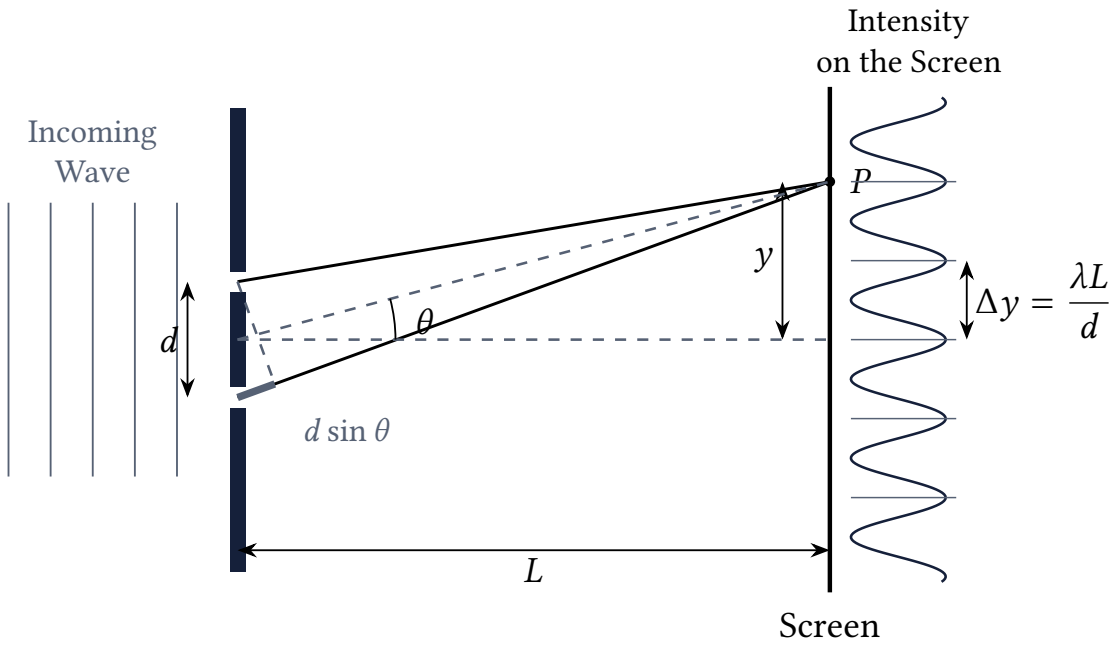


Figure 3. The double slit. Waves from the two slits, a distance d apart, reach the point P on a screen at distance L along paths that differ in length by $d \sin\theta$. Where this difference is a whole number of wavelengths the waves reinforce one another and the screen is bright; the bright bands are $\Delta y = \lambda L/d$ apart. The curve on the right is the intensity on the screen for the drawn spacing.

DERIVATION The spacing of the fringes

In every real double-slit experiment the screen is far away compared with the slit separation, $L \gg d$ (the symbol $\gg$ means much larger than), so the two paths to P are almost parallel and make the same angle θ with the axis. The path difference is then $d \sin\theta$, as in the crystal case, and the waves reinforce one another when

$$d \sin\theta = m\lambda, \qquad m = 0, \pm 1, \pm 2, \ldots$$

For small angles $\sin\theta \approx \tan\theta = y/L$, so the bright bands lie at the heights

$$y_m = \frac{m\lambda L}{d},$$

and two neighbouring bright bands are separated by $y_{m+1} - y_m = \lambda L/d$. Halfway between them, where the path difference is an odd multiple of $\lambda/2$, the two waves cancel and the screen is dark. The intensity between these points varies as $\cos^2(\pi d y/\lambda L)$, because the two waves add with a phase difference of $2\pi d \sin\theta/\lambda$ (the path difference counted in wave cycles, 2π for each wavelength), and the whole pattern is multiplied by the broad diffraction pattern of a single slit, which makes the outer fringes fainter.

KEY EQUATION Fringe spacing of the double slit

$$\Delta y = \frac{\lambda L}{d}$$

Two slits a distance d apart, illuminated by waves of wavelength λ, produce bright bands a distance Δy apart on a screen at distance L, as long as $L \gg d$ and the angles are small.

For visible light with d of a fraction of a millimetre and L of a few metres, the fringes are millimetres apart and easy to see. For electrons of a few tens of kilovolts, the energies used in electron microscopes, the wavelength is about 5 pm, a hundred thousand times smaller than that of visible light, and the slits must be a hundred thousand times finer to give the same fringes. The first person to make such slits was Claus Jönsson (1930-2024) in Tübingen. In 1961 he cut slits a fraction of a micrometre wide and about a micrometre apart into a copper foil, sent 50 kV electrons through them, magnified the pattern with electron lenses, and photographed fringes exactly like Young's.

Jönsson's photographs were taken with a dense beam, so one could still imagine that the electrons somehow interfere with one another. The decisive step was to send the electrons one at a time. Pier Giorgio Merli (1943-2008), Gian Franco Missiroli, and Giulio Pozzi did this in Bologna in 1974 and published it in 1976, and Akira Tonomura (1945-2012) and his colleagues at the Hitachi laboratory in Tokyo repeated it in 1989 with a detector that registered every single electron. Both groups used an **electron biprism** instead of two slits. A biprism is a wire thinner than a micrometre, held at a small positive voltage between two earthed plates. The electron wave that passes on the left of the wire is bent slightly to the right, the part that passes on the right is bent to the left, and behind the wire the two halves overlap at a small angle, which we call 2β. Two waves that cross at a small angle 2β produce fringes $\lambda/2\beta$ apart. This is the double-slit formula in disguise, because seen from the screen the two slits are separated by the small angle d/L, and $\Delta y = \lambda L/d$ is the wavelength divided by that angle, so the biprism formula is the double-slit formula with 2β in place of d/L. A biprism is an equivalent double slit, with the advantage that the two beams come from one wide opening and are easy to keep in step.

WORKED EXAMPLE The fringes in Tonomura's experiment

Tonomura's electrons were accelerated through 50 kV. At this voltage they move at 41 per cent of the speed of light, and the wavelength must be computed with the relativistic form of the de Broglie relation given in Section 6, which gives $\lambda = 5.4$ pm (the simple formula would give 5.5 pm). In the published geometry the two half-beams met at the angle $2\beta = 8.0 \times 10^{-6}$ radians, so the fringes behind the biprism

were

$$\Delta y = \frac{\lambda}{2\beta} = \frac{5.4 \times 10^{-12}\,\mathrm{m}}{8.0 \times 10^{-6}} = 6.7 \times 10^{-7}\,\mathrm{m}$$

apart, which is $\Delta y = 0.67\ \mu\mathrm{m}$, or 0.7 μm as the paper rounds it. Fringes this fine are far too narrow for the detector, so the electron lenses of the microscope magnified the pattern 2,000 times before it reached the detector, where the bands were 1.4 mm apart on a screen 12 mm across. The numbers are from the paper of Tonomura and his colleagues in the American Journal of Physics of 1989.

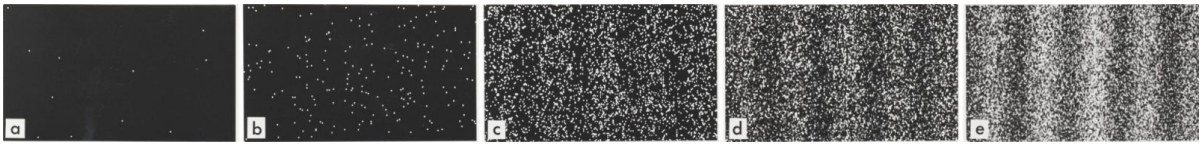


Figure 4. Tonomura's experiment of 1989: the detector screen after 11, 200, 6,000, 40,000, and 140,000 electrons (frames a to e). Every bright dot is one electron. The first frames look random, and the fringes emerge only when the dots are counted in thousands. Credit: Akira Tonomura and Hitachi, via Wikimedia Commons, CC BY-SA 3.0; the five frames of the original file are shown side by side.

Figure 4 shows what the detector recorded. Each electron arrives as one dot at one place, and nothing about the first dozen dots suggests a wave. The source delivered fewer than a thousand electrons per second, and at 41 per cent of the speed of light, which is about 120,000 km travelled every second, that is an average distance of more than 100 km between one electron and the next, in an apparatus 1.5 m long. There was never more than one electron in the microscope, so no electron could have interfered with another. Yet after twenty minutes the dots had drawn the fringes of the double slit. The wave that governs where an electron lands belongs to each electron on its own. Paul Dirac (1902-1984) put it in one sentence in his textbook of 1930, where he wrote that each photon interferes only with itself, and the same is true of electrons.

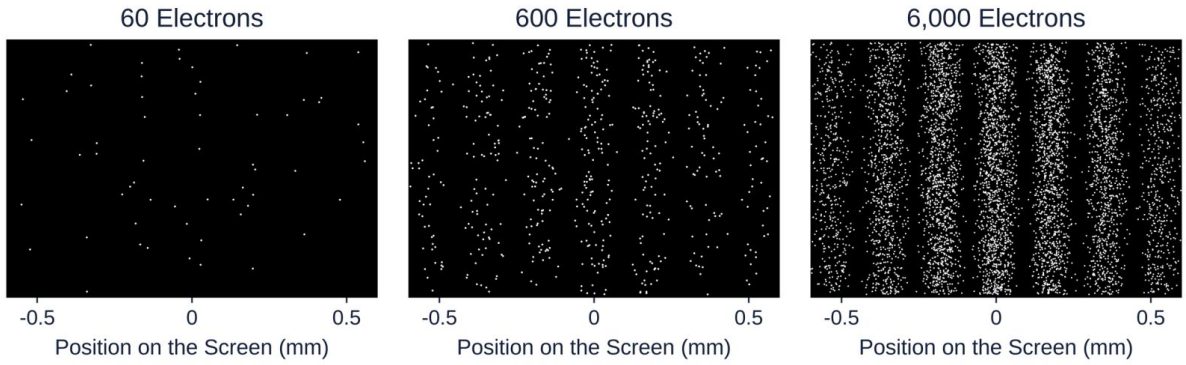


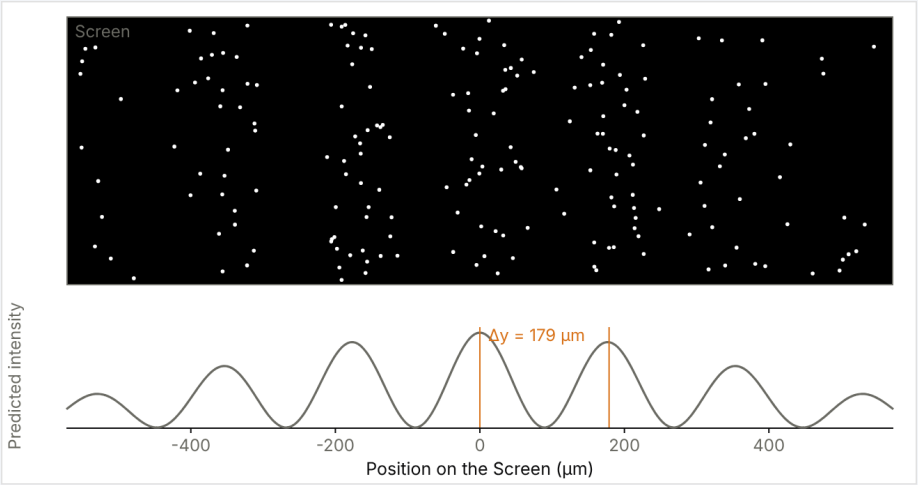
Figure 5. The build-up computed for a double slit of this unit's own geometry: electrons of 600 eV ($\lambda = 50$ pm), slits 50 nm wide and 280 nm apart, and a screen 1.0 m behind them, which gives a fringe spacing of 0.18 mm. Each dot is one arrival drawn at random from the double-slit intensity; the height of a dot carries no information. The panels show 60, 600, and 6,000 electrons.

The lesson of these pictures is the central lesson of quantum mechanics, and Unit 4 turns it into mathematics. The wave says how likely the electron is to be found at each place, and it says nothing about where the electron is before it arrives or what shape it has. Where the two waves from the slits reinforce one another, many electrons arrive; where they cancel, none do; and which of the allowed places a particular electron chooses is not determined by anything we can measure beforehand. The square of the wave amplitude is a probability, a rule that Max Born (1882-1970) formulated in 1926 and that carries his name.

For a long time the only double slits for single electrons were biprisms. In 2013 Roger Bach and his colleagues at the University of Nebraska cut two real slits, each 50 nm wide and 280 nm apart, into a thin sheet (a membrane) of silicon nitride and sent 600 eV electrons through them one at a time, which is the geometry of Figure 5. A

movable mask allowed them to close either slit. With one slit open the detector showed the broad pattern of a single slit; with both open, the fringes; and the fringes were not the sum of the two single-slit patterns. This is the thought experiment with which Richard Feynman (1918-1988) opened the quantum mechanics volume of his lectures on physics, published in 1965, performed exactly as he described it half a century later.

INTERACTIVE The double slit, one electron at a time



Add 10, 1,000, or 100,000 electrons and watch the dots turn into fringes; the grey curve below the screen is the intensity that the double-slit formula predicts for the chosen electron energy and slit separation, with the slits 50 nm wide and the screen 1 m away. Change the energy or the separation and the fringe spacing follows the formula; switch on the recording of the path and the fringes give way to the single-slit pattern, for the reason that Section 4 explains.

Click the picture or this link to try it in the online version of this unit.

CHECK YOUR UNDERSTANDING Question 3

In Tonomura's experiment the first eleven electrons landed at places that look random. What do these first dots show?

(A) The first electrons are not yet waves, and the wave nature develops only once many electrons are present

(B) Each electron lands at one point, and the probability of each point is given by the wave, which only becomes visible with many electrons

(C) The first electrons interfere with one another and only later settle into the regular pattern

(D) The detector was still warming up, and the early dots do not belong to the pattern

The solution is at the end of the unit.

EXCURSION *The most beautiful experiment*

In 2002 the magazine Physics World asked its readers to name the most beautiful experiment in physics. The winner, ahead of Galileo's falling bodies and Newton's prism, was the double slit with single electrons. The poll did not name the people who had actually done it, and that started a small historical debate. The Bologna group of Merli, Missiroli, and Pozzi had built their experiment in 1974 around a commercial electron microscope with a biprism and an image intensifier that made single electrons visible on a

television screen, and in 1976 they produced a film, Interferenza di elettroni, that showed the pattern forming dot by dot and won the physics prize at the International Festival of Scientific Films in Brussels that year. Their paper of 1976 in the American Journal of Physics filled two pages. Tonomura's experiment of 1989 used a brighter source, a field-emission tip (a very sharp needle from which a strong electric field pulls the electrons), and a detector that counted each electron with almost no losses, and his video became the standard illustration of the effect in lectures around the world. The two experiments show the same thing, and both groups deserve the credit. The voters admired the conclusion more than the apparatus: a single particle, arriving at a single point, obeys a wave.

4 Which-path information and complementarity

Since each electron arrives at one point, it is natural to ask which slit it came through. Suppose we place a detector at each slit that registers an electron passing without stopping it. Then every electron is found at one slit or the other, never at both, and the pattern on the screen changes. The fringes disappear, and what remains is the broad single-slit pattern, the same one that appears when the slits are opened one after the other. The effect does not depend on how strongly the detector disturbs the electron. It has been confirmed in experiments in which the path was recorded without any measurable push on the particle; in 1998 the group of Gerhard Rempe in Konstanz stored the path of each atom in an atom interferometer (a double slit for atoms) in the internal state of the atom itself, that is, in which of two energy levels the atom was left, and the fringes vanished all the same.

POSTULATE Complementarity

An experiment can show the interference pattern of a quantum particle, or it can reveal which path the particle took, but no experiment shows both in full. Fringes appear only when the two paths cannot be told apart by any record that exists, in the apparatus or anywhere else.

Niels Bohr (1885-1962) introduced the word **complementarity** in a lecture at Como in September 1927. Wave behaviour and particle behaviour, he argued, are both needed to describe an electron, and the experimental arrangements that show one exclude the arrangements that show the other. The statement is also quantitative, because a partial record of the path gives partial fringes. In 1996 Berthold-Georg Englert proved that the fringe contrast V (1 for fringes with fully dark gaps between them, 0 for no fringes at all) and the distinguishability D of the two paths (0 when nothing tells them apart, 1 when the record is certain) always obey $V^2 + D^2 \leq 1$, so full path knowledge, $D = 1$, forces $V = 0$. The record need not be read by anyone. It is enough that the information exists in the state of some physical system, because the fringes come from adding the two waves, and two waves that are tied to different states of a detector no longer add. Unit 6 shows the other side of the same coin, namely that any device that locates the electron at one slit gives it a momentum kick large enough to wash out the fringes, a connection that Bohr used against Einstein's objections at the Solvay conference of 1927 (Unit 8).

CHECK YOUR UNDERSTANDING Question 4

A device at the slits records which slit each electron passes through and stores the record in a memory. Nobody ever reads the memory. What appears on the screen?

(A) The fringes appear, because no one has looked at the record

(B) The fringes appear, but with half the brightness

(C) The fringes disappear, because a record of the path exists

(D) The fringes appear only for the electrons whose record was lost

The solution is at the end of the unit.

5 Interference of large molecules

Complementarity raises a practical question. A large molecule is a hot, vibrating, rotating object that constantly interacts with its surroundings, so one might expect that something always records its path and that interference stops at some size. Experiments in Vienna have pushed that limit further and further out. In 1999 Markus Arndt, Anton Zeilinger, and their colleagues sent fullerene molecules, C_{60}, sixty carbon atoms arranged like the seams of a football, through a diffraction grating. The molecules came from an oven at about 900 to 1,000 K, flew at a most probable speed (the speed shared by the largest number of molecules in the beam) of 220 m s^{-1}, and had a wavelength of 2.5 pm, about 400 times smaller than the molecule itself. The grating, a silicon nitride membrane with slits 50 nm wide and a period of 100 nm (the distance from one slit to the next), stood 1.25 m in front of a detector that ionized the molecules with a laser beam and counted them. The pattern had the diffraction orders exactly where the de Broglie wavelength put them.

WORKED EXAMPLE Why the Vienna grating needed a period of 100 nanometres

A C_{60} molecule has the mass 720 u $= 1.196 \times 10^{-24}$ kg. At 200 m s^{-1} its momentum is $p = 2.39 \times 10^{-22}$ kg m s^{-1} and its wavelength

$$\lambda = \frac{h}{p} = \frac{6.626 \times 10^{-34}\ \text{J s}}{2.39 \times 10^{-22}\ \text{kg m s}^{-1}} = 2.77 \times 10^{-12}\ \text{m} = 2.77\ \text{pm}.$$

A grating of period d sends the first diffraction order to the angle $\theta \approx \lambda/d$, which is the diffraction condition $d \sin\theta = \lambda$ with $\sin\theta \approx \theta$ for a small angle in radians. For $d = 100$ nm this is 2.77×10^{-5} radians, and 1.25 m downstream the first order is displaced by

$$y_1 = \theta L = 2.77 \times 10^{-5} \times 1.25\ \text{m} = 35\ \mu\text{m}$$

from the central beam, which is $\underline{y_1 = 35\ \mu\text{m}}$. The detecting laser beam was 8 μm wide, so orders 35 μm apart could be resolved. A grating of ordinary optical quality, with a period of 10 μm, would have placed the first order only 0.35 μm from the centre, invisible to any detector, and that is why the experiment had to wait for gratings with slits a few tens of nanometres wide.

The Vienna group then moved to a three-grating interferometer, in which a first grating prepares the wave, a laser standing wave (a laser beam reflected back on itself, whose bright and dark stripes act on the molecules like the bars of a grating) acts as the second grating, and a third grating scans the fringes, and they designed molecules for the purpose. In 2013 they showed interference of molecules of more than 10,000 u, each made of 810 atoms, and in 2019 of molecules of more than 25,000 u with about 2,000 atoms, whose de Broglie wavelength was as short as 53 fm (1 fm $= 10^{-15}$ m). The molecules are hot, but as long as they emit no photon that could reveal the path, and collide with no gas atom on the way, nothing records which slit they took and the fringes appear. When the gas pressure is raised or the molecules are heated until they radiate, the fringes fade in exactly

the way the record of the path predicts. This loss of interference through unwanted records in the surroundings is called **decoherence**, and Unit 8 returns to it.

RESEARCH SPOTLIGHT

Markus Arndt *University of Vienna*

Matter-wave interference of molecules heavier than 10,000 atomic mass units

Does the superposition principle hold for objects of any size? Markus Arndt and his group at the University of Vienna test this question by making ever larger particles interfere. For the experiment published in 2013 with Sandra Eibenberger, Stefan Gerlich, Jens Tüxen, and Marcel Mayor, the chemists in Basel built a family of porphyrin molecules fitted with long fluorinated chains, which evaporate without breaking apart and absorb little green light, the colour of the laser that forms the second grating. The heaviest member chosen for the measurement, with the formula $C_{284}H_{190}F_{320}N_4S_{12}$, has a mass of 10,123 u and consists of 810 atoms.

The molecules were evaporated at about 600 K, flew at 85 m s^{-1}, and therefore had a de Broglie wavelength of about 500 fm, ten thousand times smaller than their own diameter. They passed through three gratings of period 266 nm, the first and third cut into silicon nitride and the second formed by a standing laser wave, which shifts the phase of the matter wave without touching the molecule. Scanning the third grating across the beam and counting the molecules behind it in a mass spectrometer gave a sinusoidal fringe pattern with a contrast of 33 per cent, while a classical picture, in which the molecules are small balls that cast shadows of the gratings, predicts at most 8 per cent. The quantum prediction and the measured contrast agree over the whole range of laser powers used. The fringes come from each molecule being in a superposition of positions spread over several grating periods, with about 5,000 protons, 5,000 neutrons, and 5,000 electrons travelling along together in one delocalized wave.

Read the paper: Eibenberger, Gerlich, Arndt, Mayor, and Tüxen (2013), Matter-wave interference of particles selected from a molecular library with masses exceeding 10000 amu, Physical Chemistry Chemical Physics 15, 14696

Neutrons deserve a paragraph of their own, because they are the particle with which matter waves have been tested most precisely. A neutron is heavy and uncharged, and in a reactor beam it has a wavelength near 0.1 nm. In 1974 Helmut Rauch, Wolfgang Treimer, and Ulrich Bonse in Vienna cut an interferometer from a single perfect crystal of silicon, and since then neutron waves have been split and recombined along paths several centimetres apart. In 1975 such an interferometer showed that the Earth's gravity shifts the phase of a neutron wave by the amount that quantum mechanics predicts for a particle of the neutron's mass. In 1988 Zeilinger and his colleagues sent cold neutrons of about 2 nm wavelength through a double slit roughly a tenth of a millimetre wide and recorded a fringe pattern that agrees with the double-slit formula in every detail.

6 The electron microscope

A microscope cannot show details much smaller than the wavelength of the waves it uses, because a wave bends around any obstacle smaller than itself and carries no sharp image of it. For a light microscope the limit, worked out by Ernst Abbe (1840-1905) in 1873, is about half the wavelength, which for green light of 550 nm is roughly 200 nm, about a thousand atoms side by side. Matter waves remove this limit, because electrons accelerated through a hundred kilovolts have wavelengths of a few picometres, a hundred times smaller than an atom.



Figure 6. A replica, made in 1980, of the electron microscope with which Ernst Ruska (1906-1988) first surpassed the resolution of the light microscope in Berlin in 1933, on display in the Deutsches Museum in Munich. The column in the middle holds the magnetic lenses; the sample sits between them, and the image is viewed on a fluorescent screen at the bottom. Credit: J Brew, Wikimedia Commons, CC BY-SA 2.0.

KEY EQUATION Resolution limit of a microscope

$$\delta \approx \frac{\lambda}{2n \sin \alpha} \sim \lambda$$

The smallest distance δ that a microscope can resolve is of the order of the wavelength λ it uses; n is the refractive index of the medium and α the half-angle of the cone of waves collected by the lens. For electrons the lenses, not the wavelength, set the practical limit.

At voltages of a hundred kilovolts and more the electrons are fast enough that the simple relation $K = p^2/2m$ no longer holds. The exact relation between momentum and kinetic energy, $(pc)^2 = K^2 + 2Km_ec^2$, where $m_ec^2 = 0.511\,\text{MeV}$ is the rest energy of the electron (the energy equivalent of its mass), is given here as a fact from the theory of relativity and leads to

$$\lambda = \frac{h}{p} = \frac{hc}{\sqrt{K^2 + 2Km_ec^2}},$$

which returns to the de Broglie relation of Section 1 when $K \ll m_ec^2$.

WORKED EXAMPLE The wavelength in a 300 kilovolt electron microscope

With $K = 300\,\text{keV}$ and the product $hc = 1239.84\,\text{eV nm}$ (in these units an energy in electronvolts gives the wavelength in nanometres directly),

$$\lambda = \frac{1239.84\,\text{eV nm}}{\sqrt{(300\,\text{keV})^2 + 2 \times 300\,\text{keV} \times 511\,\text{keV}}} = \frac{1239.84\,\text{eV nm}}{6.30 \times 10^5\,\text{eV}} = 1.97 \times 10^{-3}\,\text{nm},$$

so the electrons have $\underline{\underline{\lambda = 1.97\,\text{pm}}}$, where the non-relativistic formula would give 2.24 pm. Compared with the 200 nm of a light microscope, the wavelength alone would allow a hundred thousand times finer detail. Real electron lenses are far from perfect, and the best microscopes resolve about 50 pm; in 2009 the TEAM 0.5 microscope at the Lawrence Berkeley National Laboratory resolved a spacing of 47 pm between atoms in a germanium crystal. Even so, this is 4,000 times better than light, and individual atoms are routinely imaged.

Ernst Ruska and Max Knoll (1897-1969) built the first electron microscope in Berlin in 1931, and in 1933 Ruska's instrument resolved finer detail than any light microscope could (Figure 6). Ruska received the Nobel Prize in Physics 1986 for it, more than fifty years later. The irony is that Ruska did not design his microscope as a

wave instrument. He thought of electrons as particles steered by magnetic lenses, and when he learned of de Broglie's waves after the first instrument was built, he was relieved to find that their wavelength was small enough not to spoil the images. Today the same instrument carries the biprism of Tonomura's experiment, maps the arrangement of atoms in a crystal or a virus, and in its diffraction mode repeats the experiment of Davisson and Germer every day in materials laboratories around the world.

KEY POINTS

- Every particle of momentum p has a de Broglie wavelength $\lambda = h/p$; for a particle of kinetic energy K this is $h/\sqrt{2mK}$, and for an electron accelerated through V volts it is $1.226\,\text{nm}/\sqrt{V}$.
- An electron of 54 eV has $\lambda = 0.167$ nm, a thermal neutron 0.145 nm, and a tennis ball 2×10^{-34} m; matter waves show themselves only where obstacles are as small as atoms.
- Rows of atoms a distance d apart diffract waves into the directions $d \sin\theta = n\lambda$. Davisson and Germer found the 54 V beam at 50° in 1927, and Thomson found diffraction rings from thin films in the same year.
- A double slit gives fringes $\Delta y = \lambda L/d$ apart. Electrons sent one at a time build up these fringes dot by dot, because the wave of each electron gives the probability of where it lands.
- Complementarity: an experiment that records which path a particle took shows no fringes, and fringes appear only when no record of the path exists anywhere.
- Interference has been shown for molecules of more than 25,000 u, and the short wavelength of electrons lets electron microscopes resolve distances below 50 pm.

Solutions to the questions in the text: 1-C, 2-A, 3-B, 4-C. The online version of this unit explains each answer.